DIFFRACTION OF A PLANE WAVE AT THE JUNCTION OF AN IDEALLY CONDUCTIVE AND A DIELECTRIC WEDGE

(DIFRAKTSIIA PLOSKOI VOLNY NA STYKE IDEAL'NO PROVODIASHCHEGO I DIELEKTRICHESKOGO KLIN'EV)

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S.S. KALMYKOVA
(Khar'kov)

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We consider a rectangular wedge ($x > 0$, $z > 0$) filled with an anisotropic dielectric ($\varepsilon_x = \varepsilon_\perp$, $\varepsilon_z = \varepsilon_\parallel$) in contact with an ideally conductive rectangular wedge ($x > 0$, $z < 0$). A plane wave with the components (superscript $\rightarrow$)

$$\mathbf{H}_y^{\rightarrow} = \exp\,[ikz\cos\varphi_0 + ikx\sin\varphi_0]$$

$$\mathbf{E}_z^{\rightarrow} = -\sin\varphi_0\exp\,[ikz\cos\varphi_0 + ikx\sin\varphi_0], \qquad k = \omega/c, \qquad \operatorname{Im} k > 0 \tag{1}$$

coming from the side of the half-space $x < 0$ (vacuum) strikes the surface of the ideally conductive wedge ($x = 0$), $z < 0$) at an angle φ_0.

Let us find the equation that describes the fields scattered at the wedge boundary. We represent the total field in the region $x < 0, -\infty < z < \infty$ as the sum of the incident and reflected waves (superscript $\leftarrow$) (the solution corresponding to the ideally conductive half-space) plus the field that decays due to the presence of the wedge junction ($z = 0$) which we shall attempt to find as an integral of plane waves [1 and 2]

$$\mathbf{H}_y^{\leftarrow}(x < 0) = \exp\,[ikz\cos\varphi_0 - ikx\sin\varphi_0], \quad H_y^{*} = \int_{-\infty}^{+\infty} h(\tau)\, e^{i\tau z + vx}\, d\tau \tag{2}$$

$$\mathrm{E}_z^{\leftarrow} = \sin\varphi_0\exp\,[ikz\cos\varphi_0 - ikx\sin\varphi_0], \qquad E_z^{*} = \int_{-\infty}^{+\infty} \frac{iv}{k}\, h(\tau)\, e^{i\tau z + vx}\, d\tau$$

$$v = (\tau^2 - k^2)^{1/2}$$

(The amplitude and phase of the scattered wave (superscript *) guarantee fulfillment of the boundary condition $E_z^{\rightarrow} + E_z^{\leftarrow} = 0$ with $x = 0$ as $z \to -\infty$ when the scattered field is equal to zero).

The scattered field in the dielectric wedge can also be represented as a superposition of plane waves

$$\mathbf{H}_y^{*} = \int_{-\infty}^{+\infty} H^\circ(\tau)\, e^{i\tau z - \beta x} d\tau, \qquad \beta = \left[\frac{\varepsilon_\parallel}{\varepsilon_\perp}(\tau^2 - k^2\varepsilon_\perp)\right]^{1/2}$$

$$\mathbf{E}_z^{*} = -\int_{-\infty}^{+\infty} \frac{i\beta}{k\varepsilon_\parallel} H^\circ(\tau)\, e^{i\tau z - \beta x} d\tau, \qquad \mathbf{E}_x^{*} = \int_{-\infty}^{+\infty} \frac{\tau}{k\varepsilon_\perp} H^\circ(\tau)\, e^{i\tau z - \beta x} d\tau \tag{3}$$

Fields (1) to (3) satisfy Maxwell's conditions. Requiring fulfillment of the boundary conditions, we obtain equations for determining the unknown functions $h(\tau)$ and $H^\circ(\tau)$. These boundary conditions consist of the continuity of the complete fields at the surface $x = 0$, $z > 0$ and in the requirement that the tangential components of the scattered electric field vanish at the faces of the ideally conductive wedge

$$\begin{gathered}
\int_{-\infty}^{+\infty} [H^\circ(\tau) - h(\tau)]\, e^{i\tau z}\, d\tau = 2\exp(ikz\cos\varphi_0), \quad z > 0, \quad x = 0 \\
\int_{-\infty}^{+\infty} [Z_1(\tau)\, H^\circ(\tau) + Z_2(\tau)\, h(\tau)]\, e^{i\tau z}\, d\tau = 0 \qquad \left(Z_1(\tau) = \frac{i\beta}{k\varepsilon_\parallel}, \quad Z_2(\tau) = \frac{iv}{k}\right) \\
\int_{-\infty}^{+\infty} Z_2(\tau)\, h(\tau)\, e^{i\tau z}\, d\tau = 0, \quad z < 0, \quad x = 0
\end{gathered} \tag{4}$$

$$\int_{-\infty}^{+\infty} \frac{i\tau}{k\varepsilon_\perp} H^\circ(\tau)\, e^{-\beta x} d\tau = 0, \quad z = 0, \quad x > 0 \tag{5}$$

On the basis of conditions (4), we can use the Rapoport lemma [3] to express $h(\tau)$ and $H^\circ(\tau)$ in terms of the boundary values at the contour $\operatorname{Im}\tau = 0$ of functions analytic in the upper (superscript +) and lower (superscript –) half-planes of the complex variable τ

$$\begin{gathered}
h(\tau) = \frac{1}{Z_2(\tau)}\, \xi^-(\tau) \\
H^\circ(\tau) = \frac{1}{Z_1(\tau)}\, [\psi^+(\tau) - \xi^-(\tau)] \\
H^\circ(\tau) - h(\tau) = \varphi^+(\tau) + \frac{1}{\pi i(\tau - k\cos\varphi_0)}
\end{gathered} \tag{6}$$

Eliminating $h(\tau)$ and $H^\circ(\tau)$ from (6), we obtain the boundary value problem

$$\varphi^+(\tau) = \frac{1}{Z_1(\tau)}\, \psi^+(\tau) - \frac{\Delta(\tau)}{Z_1 Z_2}\, \xi^-(\tau) - \frac{1}{\pi i(\tau - k\cos\varphi_0)}, \qquad \Delta = Z_1 + Z_2 \tag{7}$$

This boundary value problem contains three unknown functions. The relation between two of them can be obtained from condition (5), which requires that the function $H^\circ(\tau)$ be even

$$\psi^+(\tau) + \xi^-(-\tau) = \xi^-(\tau) + \psi^+(-\tau) \tag{8}$$

Sommerfeld's condition [1 and 4] which consists in the requirement that the field H_y be finite near the edge $z = 0$, $x = 0$ of the conductive wedge, makes the left- and right-hand sides of (8) vanish. We therefore have

$$\varphi^+(\tau) = -\frac{1}{Z_1(\tau)}\, \xi^-(-\tau) - \frac{\Delta(\tau)}{Z_1(\tau)\, Z_2(\tau)}\, \xi^-(\tau) - \frac{1}{\pi i(\tau - k\cos\varphi_0)} \tag{9}$$

A relation similar to (9) for a problem involving an ideally conductive semi-infinite strip of finite thickness in a waveguide with a homogeneous filler and ideally conductive walls was first obtained by Johnson [5], who also suggested a procedure for reducing such a relation to an infinite system of linear algebraic equations (provided that the coefficients in the problem are integer functions).

We now show that relation (9) is equivalent to Fredholm's equation of the second kind. Let us note first that an expression of the type (9) is equivalent to a system of two boundary value problems for two piecewise-continuous functions (the second relation can be obtained by replacing τ by $-\tau$).

Since the coefficients in (9) are even functions of τ , we obtain

$$\frac{1}{\pi i}\int\limits_{-\infty}^{+\infty}\frac{\xi(\tau')\,d\tau'}{\tau'-\tau}+\frac{Z_1(\tau)\,Z_2(\tau)}{2\Delta(\tau)\,\pi i}\int\limits_{-\infty}^{+\infty}\left[\frac{1}{Z_2(\tau')}-\frac{1}{Z_2(\tau)}\right]\frac{\xi(\tau')\,d\tau'}{\tau'-\tau}=$$
$$=-\frac{2Z_1(\tau)\,Z_2(\tau)\,k\cos\varphi_0}{\Delta(\tau)\,\pi i\,(\tau^2-k^2\cos^2\varphi_0)} \tag{10}$$

where

$$\xi(\tau)=\xi^-(-\tau)-\xi^-(\tau),\qquad \xi^-(-\tau)+\xi^-(\tau)=\frac{1}{\pi i}\int\limits_{-\infty}^{+\infty}\xi\,\frac{(\tau')\,d\tau'}{\tau'-\tau}$$

The problem of diffraction of a plane wave at the junction of an ideally conductive and an anisotropic dielectric rectabgular wedge reduces to the solution of a singular integral equation with a Cauchy kernel. Equation (10) can be investigated by means of the well-known theory of such equations (e.g. see the monograph by Muskhelishvili [6]).

Since $\operatorname{Im} k \neq 0$, it follows that $\Delta(\tau)$ does not vanish on the real axis. The index of this equation is equal to zero. Equation (10) is therefore equivalent to the Fredholm equation:

$$\xi(\tau)=-\frac{2k\cos\varphi_0}{(\pi i)^2}\int\limits_{-\infty}^{+\infty}\frac{Z_1(\tau')\,Z_2(\tau')\,d\tau'}{\Delta(\tau')\,(\tau'^2-k^2\cos^2\varphi_0)\,(\tau'-\tau)}-$$
$$-\frac{1}{2(\pi i)^2}\int\limits_{-\infty}^{+\infty}\frac{Z_1(\tau')\,Z_2(\tau')\,d\tau'}{\Delta(\tau')\,(\tau'-\tau)}\int\limits_{-\infty}^{+\infty}\left[\frac{1}{Z_2(u)}-\frac{1}{Z_2(\tau')}\right]\frac{\xi(u)\,du}{u-\tau'} \tag{11}$$

The latter equation can be solved numerically, and in the presence of a small parameter (e.g. $|\varepsilon_0|^{-1}\ll 1$) by the method of successive approximations.

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